4.1. Napíšte rovnicu dotyčnice a normály ku grafu funkcie *f* v bode *T*.

a) $f: y = x^2 - 2x + 2,\ T = [0, ?]$

b) $f: y = 2x - x^2,\ T = [1, ?]$

c) $f: y = 1 - \dfrac{1}{x+1},\ T = [0, ?]$

d) $f: y = \sqrt[3]{x+4},\ T = [-3, ?]$

e) $f: y = \dfrac{2x}{x+1},\ T = [0, ?]$

f) $f: y = x + \sqrt{4-x},\ T = [3, ?]$

g) $f: y = (x-1)\cdot e^x,\ T = [1, ?]$

h) $f: y = \ln \sin x,\ T = \left[\dfrac{\pi}{2}, ?\right]$

4.2. Napíšte rovnicu dotyčnice ku grafu funkcie *f*, ktorá zviera s osou *x* uhol 45°.

a) $f: y = 2\cdot\sqrt{x^2+3}$

b) $f: y = \mathrm{arctg}2x$

4.3. Napíšte rovnicu dotyčnice a normály ku grafu funkcie *f*, ak dotyčnica *t* je rovnobežná s danou priamkou *p*.

a) $f: y = \ln(x+1),\ p: x - y + 2 = 0$

b) $f: y = 3 - 2\cdot e^{\frac{x}{2}},\ p: 2x + 2y - 3 = 0$

c) $f: y = x^3 - x,\ p: 2x - y = 0$

d) $f: y = \dfrac{2x-1}{2-x},\ p: 3x - y = 0$

4.4. Dané sú funkcie celkových nákladov a príjmov. Vypočítajte hodnoty marginálnych nákladov, príjmov a zisku pre danú úroveň produkcie *x* a výsledky ekonomicky interpretujte.

a) $C(x) = 600 + 20x,\ R(x) = 30x,\ x = 50$

b) $C(x) = x^2 - 6x + 25,\ R(x) = 25x - 2x^2,\ x = 5$

c) $C(x) = 50 + 3x - 0{,}01x^2,\ R(x) = 2{,}5x - 0{,}005x^2,\ x = 25$

d) $C(x) = \dfrac{x^3}{3} - 0{,}5x^2 + 40,\ R(x) = 150x - 0{,}2x^2,\ x = 10$

e) $C(x) = 100\cdot e^{0{,}01x},\ R(x) = 10x\cdot e^{0{,}01x},\ x = 100$

4.5. Vypočítajte marginálny dopyt pre jednotkovú cenu *p*, ak je daná funkcia dopytu *d*. Výsledky ekonomicky interpretujte.

a) $d: q = 56 - 2p,\ p = 6$

b) $d: q = 100 - p^2,\ p = 4$

c) $d: q = 1 + 3e^{-\frac{p}{3}},\ p = 3$

d) $d: q = \dfrac{60}{p+3} - 2,\ p = 2$

4.6. Daná je dopytová funkcia d. Vypočítajte elasticitu dopytu pre jednotkovú cenu p. Výsledky ekonomicky interpretujte.

a) $d: q = 27 - 3p,\ p = 3$

b) $d: q = \dfrac{32 - p^2}{2},\ p = 4$

c) $d: q = \dfrac{12}{p} - 3,\ p = 2$

d) $d: q = 80 - 30\sqrt{p},\ p = 4$

4.1

a) $f: y = x^2 - 2x + 2 \quad ; \quad T[0; ?]$

~ najprv zistíme y-ovú súradnicu bodu T:

$y = 0^2 - 2\cdot 0 + 2 = 2$ (dosadili sme x-ovú súradnicu do predpisu funkcie)

$T[0; 2]$

~ potom zistíme deriváciu funkcie:

$f': y' = (x^2 - 2x + 2)' = \underbrace{2x - 2}$

Vzorce:

~ rovnica dotyčnice v bode $A[a, f(a)]$: $\boxed{y = f(a) + f'(a)\cdot(x-a)}$

~ rovnica normály v bode $A[a, f(a)]$: $\boxed{y = f(a) + \frac{-1}{f'(a)}\cdot(x-a)}$

~ nakoniec dosadíme do vzorcov:

dotyčnica: $y = \overset{\text{y-ová súr. T}}{2} + \overbrace{(2\cdot 0 - 2)}^{f' \text{ v x-ovej súr. T}}\cdot(x - \overset{\text{x-ová súr. T}}{0}) = 2 + (-2)\cdot x = 2 - 2x = -2x + 2$

$y = -2x + 2$

normála: $y = 2 + \frac{-1}{2\cdot 0 - 2}\cdot(x - 0) = 2 + \frac{-1}{-2}\cdot x = 2 + \frac{1}{2}x$

$y = \frac{1}{2}x + 2$

b) $f: y = 2x - x^2 \quad ; \quad T = [1; ?]$

$y = 2\cdot 1 - 1^2 = 1 \Rightarrow T[1; 1]$

$f': y' = \underbrace{2 - 2x}$

dotyčnica: $y = 1 + (2 - 2\cdot 1)\cdot(x - 1) = 1 + 0\cdot(x-1)$

$y = 1$

normála: $y = 1 + \frac{-1}{2 - 2\cdot 1}\cdot(x-1) = 1 - \frac{1}{0}\cdot(x-1) \rightsquigarrow \frac{1}{0}$ nemá zmysel

normála $\nexists$

c) $f: y = 1 - \frac{1}{x+1}$; $T = [0; ?]$

$y(0) = 1 - \frac{1}{0+1} = 0 \Rightarrow T[0;0]$

$y' = -\frac{-(x+1)'}{(x+1)^2} = \frac{+1}{(x+1)^2}$

dotyčnica:

$y = 0 + \frac{1}{(0+1)^2}\cdot(x-0) = \frac{1}{1}\cdot x$

$y = x$

normála:

$y = 0 + \frac{-1}{\frac{1}{(0+1)^2}}\cdot(x-0) = \frac{-1}{1}\cdot(x-0) = -x$

$y = -x$

d) $f: y = \sqrt[3]{x+4}$; $T = [-3; ?]$

$y(-3) = \sqrt[3]{-3+4} = 1 \Rightarrow T[-3;1]$

$y' = \left[(x+4)^{\frac{1}{3}}\right]' = \frac{1}{3}(x+4)^{-\frac{2}{3}}\cdot(x+4)' = \frac{1}{3}\cdot\frac{1}{\sqrt[3]{(x+4)^2}}\cdot 1 = \frac{1}{3\cdot\sqrt[3]{(x+4)^2}}$

dotyčnica:

$y = 1 + \frac{1}{3\cdot\sqrt[3]{(-3+4)^2}}\cdot(x+3) = 1 + \frac{1}{3}\cdot(x+3)$

$= 1 + \frac{1}{3}x + 1$

$y = \frac{1}{3}x + 2$

normála: $y = 1 + \frac{-1}{\frac{1}{3}}\cdot(x+3) = 1 - 3\cdot(x+3)$

$= 1 - 3x - 9$

$y = -3x - 8$

e) $f: y = \frac{2x}{x+1}$; $T[0; ?]$

$y(0) = \frac{2\cdot 0}{0+1} = 0 \Rightarrow T[0;0]$

$f' = \frac{2\cdot(x+1) - 2x\cdot 1}{(x+1)^2} = \frac{2x+2-2x}{(x+1)^2} = \frac{2}{(x+1)^2}$

4.1

e) dotyčnica:

$$y = 0 + \frac{2}{(0+1)^2}\cdot(x-0) = \frac{2}{1}\cdot x \Rightarrow \underline{\underline{y = 2x}}$$

normála:

$$y = 0 + \frac{-1}{\frac{2}{1}}\cdot(x-0) = -\frac{1}{2}x \Rightarrow \underline{\underline{y = -\frac{1}{2}x}}$$

f) $f: y = x + \sqrt{4-x}\ ;\ T = [3; ?]$

$$y(3) = 3 + \sqrt{4-3} = 4 \Rightarrow T[3;4]$$

$$y' = 1 + \left[(4-x)^{\frac{1}{2}}\right]' = 1 + \frac{1}{2}(4-x)^{-\frac{1}{2}}\cdot(-1) = 1 + \frac{1}{2}\cdot\frac{1}{\sqrt{4-x}}\cdot(-1) = \underbrace{1 - \frac{1}{2\cdot\sqrt{4-x}}}$$

dotyčnica: $y = 4 + 1 - \frac{1}{2\cdot\sqrt{4-3}}\cdot(x-3) =$

$$= 5 - \frac{1}{2}\cdot(x-3) = 5 - \frac{1}{2}x + \frac{3}{2}$$

$$\underline{\underline{y = -\frac{1}{2}x + \frac{13}{2}}}$$

normála:

$$y = 4 + \frac{-1}{1-\frac{1}{2}}\cdot(x-3) = 4 - 2\cdot(x-3)$$

$$\underline{\underline{y = -2x + 10}}$$

g) $f: y = (x-1)\cdot e^x\ ;\ T = [1; ?]$

$$y(1) = 0\cdot e = 0 \Rightarrow T = [1;0]$$

$$f' = 1\cdot e^x + (x-1)\cdot e^x = e^x\cdot(1+x-1) = \underbrace{x\cdot e^x}$$

dotyčnica:

$$y = 0 + 1\cdot e^1\cdot(x-1) = e\cdot(x-1) = ex - e$$

$$\underline{\underline{y = ex - e}}$$

normála:

$$y = 0 + \frac{-1}{e}\cdot(x-1) = -\frac{1}{e}x + \frac{1}{e}$$

$$\underline{\underline{y = -\frac{1}{e}x + \frac{1}{e}}}$$

4.1

h) $f: y = \ln(\sin x)$; $T = [\frac{\pi}{2}; ?]$

$y(\frac{\pi}{2}) = \ln(\sin\frac{\pi}{2}) = \ln 1 = 0 \Rightarrow T[\frac{\pi}{2}; 0]$

$y' = \frac{1}{\sin x} \cdot \cos x = \frac{\cos x}{\sin x} = \underline{\cot g\, x}$

dotyčnica: $y = 0 + \cot g \frac{\pi}{2} \cdot (x - \frac{\pi}{2}) = 0 \cdot (x - \frac{\pi}{2}) = 0$

$\underline{\underline{y = 0}}$

normála: $y = 0 + \frac{-1}{0} \cdot (x - \frac{\pi}{2})$

$\underline{\underline{\text{normála } \nexists}}$

4.2

a) $f: y = 2 \cdot \sqrt{x^2+3}$; $\alpha = 45°$

$\boxed{\text{platí: } f'(x_0) = \mathrm{tg}\,\alpha}$

$y' = 2 \cdot [(x^2+3)^{\frac{1}{2}}]' = 2 \cdot \frac{1}{2} \cdot (x^2+3)^{-\frac{1}{2}} \cdot 2x =$

$= 2x \cdot \frac{1}{\sqrt{x^2+3}} = \underline{\frac{2x}{\sqrt{x^2+3}}}$

$\frac{2 \cdot x_0}{\sqrt{x_0^2+3}} = \mathrm{tg}\, 45° = 1 \quad / \cdot \sqrt{x_0^2+3}$

$2x_0 = \sqrt{x_0^2+3} \quad /^2$

$4x_0^2 = x_0^2 + 3$

$3x_0^2 = 3$

$x_0^2 = 1$

$\underline{x_0 = \pm 1}$ ⇝ dosadíme do pôvodného predpisu aby sme zistili y_0:

$y_0 = 2 \cdot \sqrt{x_0^2+3}$

$y_0 = 2 \cdot \sqrt{4} = 4 \Rightarrow T_1 = [-1; 4] \wedge T_2 = [1; 4]$ → ak ale dosadíme $x_0 = -1$ späť, zistíme že vzťah neplatí $\Rightarrow T = [1; 4]$

dotyčnica:

$y = 4 + \frac{2 \cdot 1}{\sqrt{1^2+3}} \cdot (x-1) = 4 + \frac{2}{2} \cdot (x-1) = 4 + x - 1$

$\underline{\underline{y = x + 3}}$

normála:

$y = 4 + \frac{-1}{\frac{2}{2}} \cdot (x-1) = 4 - x + 1$

$\underline{\underline{y = -x + 5}}$

4.2

b) $f: y = \operatorname{arctg} 2x$; $\alpha = 45^\circ \Rightarrow \boxed{\operatorname{tg}\alpha = 1}$

$$y' = (\operatorname{arctg} 2x)' = \frac{1}{1+(2x)^2}\cdot 2 = \boxed{\frac{2}{1+4x^2}}$$

$$\frac{2}{1+4\cdot x_0^2} = 1 \quad /\cdot(1+4x_0^2)$$

$$2 = 1+4x_0^2$$

$$4x_0^2 = 1$$

$$x_0^2 = \frac{1}{4}$$

$\boxed{x_0 = \pm\frac{1}{2}}$ → keď dosadíme späť, zistíme, že obe možnosti vyhovujú

zistíme y_0:

$$y_0 = \operatorname{arctg} 2\cdot\left(\pm\frac{1}{2}\right) = \operatorname{arctg}(\pm 1)$$

ak $x_0 = -\frac{1}{2} \Rightarrow y_0 = \operatorname{arctg}(-1) = -\frac{\pi}{4}$

$x_0 = \frac{1}{2} \Rightarrow y_0 = \operatorname{arctg} 1 = \frac{\pi}{4}$

$T_1\left[-\frac{1}{2}; -\frac{\pi}{4}\right]$

$T_2\left[\frac{1}{2}; \frac{\pi}{4}\right]$

1\. dotyčnica:

$$y = \frac{\pi}{4} + \frac{2}{1+4\cdot\left(\frac{1}{2}\right)^2}\cdot\left(x-\frac{1}{2}\right) = \frac{\pi}{4} + \frac{2}{2}\cdot\left(x-\frac{1}{2}\right)$$

$$y = x - \frac{1}{2} + \frac{\pi}{4}$$

1\. normála: $y = \frac{\pi}{4} + \frac{-1}{\frac{2}{2}}\cdot\left(x-\frac{1}{2}\right) = \frac{\pi}{4} - x + \frac{1}{2}$

$$y = -x + \frac{1}{2} + \frac{\pi}{4}$$

2\. dotyčnica:

$$y = -\frac{\pi}{4} + \frac{2}{1+4\cdot\left(-\frac{1}{2}\right)^2}\cdot\left(x+\frac{1}{2}\right) = -\frac{\pi}{4} + \frac{2}{2}\cdot\left(x+\frac{1}{2}\right)$$

$$y = x + \frac{1}{2} - \frac{\pi}{4}$$

2\. normála:

$$y = -\frac{\pi}{4} + \frac{-1}{\frac{2}{2}}\cdot\left(x+\frac{1}{2}\right) = -\frac{\pi}{4} - x - \frac{1}{2}$$

$$y = -x - \frac{1}{2} - \frac{\pi}{4}$$

4.3

a) $f: y=\ln(x+1)$; $p: y=x+2$; $t \parallel p$ → keďže sú dané priamky rovnobežné, majú rovnaké smernice, t.j. číslo pred „x" v danom predpise $y=x+2=1\cdot x+2$ a teda smernica dotyčnice bude tiež $=1$

$$y'=\frac{1}{x+1}$$

$$\frac{1}{x_0+1}=1$$

$$1=x_0+1$$

$$x_0=0$$

$$y_0=\ln(x_0+1)=\ln(0+1)=0 \Rightarrow T[0;0]$$

dotyčnica:

$$y=0+\frac{1}{0+1}\cdot(x-0)=x \Rightarrow y=x$$

normála:

$$y=0+\frac{-1}{\frac{1}{1}}\cdot(x-0) \Rightarrow y=-x$$

b) $f: y=3-2\cdot e^{\frac{x}{2}}$; $p: 2x+2y-3=0$; $p \parallel t$

$$2y=-2x+3$$

$$y=-x+\frac{3}{2}$$

$$y'=-2\cdot e^{\frac{x}{2}}\cdot\frac{1}{2}=-e^{\frac{x}{2}}$$

$$-e^{\frac{x_0}{2}}=-1$$

$$e^{\frac{x_0}{2}}=1 \quad [e^0=1]$$

$$\frac{x_0}{2}=0$$

$$x_0=0 \rightarrow y_0=3-2\cdot e^{\frac{0}{2}}=1 \Rightarrow T=[0;1]$$

dotyčnica:

$$y=1+\left(-e^{\frac{0}{2}}\right)\cdot(x-0)=1-1\cdot x \Rightarrow y=-x+1$$

normála:

$$y=1+\frac{-1}{-1}\cdot(x-0)=1+x \Rightarrow y=x+1$$

4.3

c) $f: y = x^3 - x$; $p: y = 2x$; $t \parallel p$

$\underline{y' = 3x^2 - 1}$

$3x_0^2 - 1 = 2$

$3x_0^2 = 3$

$x_0^2 = 1$

$\underline{x_0 = \pm 1}$

$x_0 = -1 \Rightarrow y_0 = (-1)^3 - (-1) = -1 + 1 = 0$

$x_0 = 1 \Rightarrow y_0 = 1^3 - 1 = 0$

$\Rightarrow$ $T_1[-1; 0]$ $T_2[+1; 0]$

1. dotyčnica:

$y = 0 + (3 \cdot (-1)^2 - 1) \cdot (x+1) = 2 \cdot (x+1) \Rightarrow \underline{\underline{y = 2x + 2}}$

1. normála:

$y = 0 + \frac{-1}{2} \cdot (x+1) = -\frac{1}{2} \cdot (x+1) \Rightarrow \underline{\underline{y = -\frac{1}{2}x - \frac{1}{2}}}$

2. dotyčnica:

$y = 0 + (3 \cdot 1^2 - 1) \cdot (x-1) = 2 \cdot (x-1) \Rightarrow \underline{\underline{y = 2x - 2}}$

2. normála:

$y = 0 + \frac{-1}{2} \cdot (x-1) = -\frac{1}{2} \cdot (x-1) \Rightarrow \underline{\underline{y = -\frac{1}{2}x + \frac{1}{2}}}$

d) $f: y = \frac{2x-1}{2-x}$; $p: y = 3x$; $p \parallel t$

$y' = \frac{2 \cdot (2-x) - (2x-1) \cdot (-1)}{(2-x)^2} = \frac{4 - 2x + 2x - 1}{(2-x)^2} = \underline{\frac{3}{(2-x)^2}}$

$\frac{3}{(2-x_0)^2} = 3$

$3 = 3 \cdot (2-x_0)^2$

$(2-x_0)^2 = 1$

$(2-x_0)^2 - 1 = 0$ $[a^2 - b^2 = (a+b) \cdot (a-b)]$

$(2 - x_0 + 1) \cdot (2 - x_0 - 1) = 0$ $x_{01} = 3$

$(3 - x_0)(1 - x_0) = 0 \Rightarrow x_{02} = 1$

$x_0 = 3 \Rightarrow y_0 = \frac{2 \cdot 3 - 1}{2 - 3} = \frac{5}{-1} = -5$

$x_0 = 1 \Rightarrow y_0 = \frac{2 \cdot 1 - 1}{2 - 1} = \frac{1}{1} = 1$

$\Rightarrow$ $T_1[3; -5]$ $T_2[1; 1]$

1. dotyčnica: $y = -5 + \frac{3}{(2-3)^2} \cdot (x-3) = -5 + 3(x-3)$

$\underline{\underline{y = 3x - 14}}$

1. normála: $y = -5 + \frac{-1}{3} \cdot (x-3)$

$\underline{\underline{y = -\frac{1}{3}x - 4}}$

2. dotyčnica: $y = 1 + \frac{3}{(2-1)^2} \cdot (x-1) = 1 + 3 \cdot (x-1)$

$\underline{\underline{y = 3x - 2}}$

2. normála: $y = 1 + \frac{-1}{3} \cdot (x-1)$

$\underline{\underline{y = -\frac{1}{3}x + \frac{4}{3}}}$