**6.1. Nájdite intervaly konvexnosti a konkávnosti funkcie $f$. Určte inflexné body funkcie, ak existujú.**

a) $f: y = x^3 - 9x^2 + 1$

b) $f: y = x^4 - 2x^3 - 7$

c) $f: y = x^4 + 4x^3 - 18x^2 + 3x + 2$

d) $f: y = x^4 - x^5$

e) $f: y = \dfrac{x^6}{6} - \dfrac{x^5}{4} + 3$

f) $f: y = 3x - (4 - x)^5$

g) $f: y = x^4 + 2x^3 + 6x^2$

h) $f: y = x + \dfrac{1}{x}$

i) $f: y = 3x + \dfrac{1}{2x^2}$

j) $f: y = \dfrac{3x^2}{1 - x}$

k) $f: y = \dfrac{x^2 + x + 21}{x + 2}$

l) $f: y = \dfrac{2x}{1 + x^2}$

m) $f: y = \dfrac{x}{1 - x^2}$

n) $f: y = \dfrac{x^2}{16 - x^2}$

**6.2. Nájdite intervaly konvexnosti a konkávnosti funkcie $f$. Určte inflexné body funkcie, ak existujú.**

a) $f: y = \dfrac{x^2 + 1}{x^2 - 1}$

b) $f: y = \dfrac{1}{x^3} + \dfrac{1}{x^2}$

c) $f: y = \dfrac{1}{x^3} - \dfrac{6}{x}$

d) $f: y = \left(\dfrac{1}{2} + \dfrac{1}{x}\right)^2$

e) $f: y = 3x - \sqrt{x - 3}$

f) $f: y = 4 + \sqrt[3]{x^2}$

g) $f: y = \dfrac{2x}{\sqrt{x^2 + 1}}$

h) $f: y = \dfrac{x}{\sqrt{x^3 + 1}}$

**6.3. Nájdite intervaly konvexnosti a konkávnosti funkcie $f$. Určte inflexné body funkcie, ak existujú.**

a) $f: y = x \cdot e^{-x}$

b) $f: y = e^{-x^2}$

c) $f: y = x \cdot e^{-x^2}$

d) $f: y = x^2 \cdot e^{-x}$

e) $f: y = x \cdot e^{\frac{1}{x}}$

f) $f: y = e^{4 - \frac{x^2}{2}}$

g) $f: y = e^{1 - \frac{x^3}{3}}$

h) $f: y = e^{2x} - 8e^x + 5x$

i) $f: y = (2 - x^2) \cdot e^{-x}$

j) $f: y = \dfrac{e^x}{x}$

# Teória: Aplikácie derivácie funkcie:

**1. Monotónnosť:** pre všetky $x \in D(f)$ také, že $f'(x) > 0$ platí, že funkcia rastie

pre $\forall x \in (a,b)$: $f'(x) < 0 \Rightarrow f(x)$ na $(a,b)$ klesá

**2. Konvexnosť, konkávnosť:** pre $\forall x \in (a,b)$: $f''(x) > 0 \Rightarrow f(x)$ je na $(a,b)$ konvexná

pre $\forall x \in (a,b)$: $f''(x) < 0 \Rightarrow f(x)$ je na $(a,b)$ konkávna

**3. Lokálne extrémy:** pre $\forall x_0 \in D(f)$: $f'(x_0) = 0 \Rightarrow f(x)$ má v bode $x_0$ lokálny extrém

$x_0$ sa nazýva stacionárny bod

ak $f''(x_0) > 0 \Rightarrow f(x)$ má v $x_0$ lokálne minimum

ak $f''(x_0) < 0 \Rightarrow f(x)$ má v $x_0$ lokálne maximum

**4. Inflexné body:** pre $\forall x_0 \in D(f)$: $f''(x_0) = 0 \Rightarrow f(x)$ sa mení v $x_0$ z konvexnej na konkávnu (alebo opačne)

$x_0$ sa nazýva inflexný bod

5.3 m) $f: y = \frac{2}{x} + \ln x^2$ ; $\boxed{D(f) = (-\infty; 0) \cup (0; \infty)}$

$$y' = -2x^{-2} + \frac{1}{x^2} \cdot 2x = \frac{-2}{x^2} + \frac{2}{x} = \frac{-2+2x}{x^2} = \frac{2x-2}{\underbrace{x^2}_{>0}} > 0 \Leftrightarrow 2x-2>0 \;/:2$$

$x-1>0$

$\underbrace{x>1}$ $\Rightarrow$ na $(1;\infty)$ rastie

na $(-\infty;0)$ a na $(0;1)$ klesá

n) $f: y = \frac{x}{\ln x}$ ; $D(f)$: $\underbrace{x>0} \wedge \ln x \neq 0$

$\underbrace{x \neq 1}$ $\Rightarrow$ $\boxed{D(f) = (0;1) \cup (1;\infty)}$

$$y' = \frac{1 \cdot \ln x - x \cdot \frac{1}{x}}{\ln^2 x} = \frac{\ln x - 1}{\underbrace{\ln^2 x}_{>0}} > 0 \Leftrightarrow \ln x - 1 > 0$$

$\ln x > 1$

$\underbrace{x>e}$ $\Rightarrow$ na $(e;\infty)$ rastie

na $(0;1)$ a na $(1;e)$ klesá

6.1 a) $f: y = x^3 - 9x^2 + 1$

$y' = 3x^2 - 18x$

$y'' = 6x - 18 = 0$

$6x = 18$

$x = 3 \rightarrow$ inflexný bod

$y'' > 0$

$6x - 18 > 0$

$x > 3 \Rightarrow$ funkcia konvexná, inak konkávna

$\Rightarrow$ na $(3;\infty)$ konvexná

na $(-\infty;3)$ konkávna

$x_0 = 3$ je inflexný bod

6.1

b) $f: y = x^4 - 2x^3 - 7$

$y' = 4x^3 - 6x^2$

$y'' = 12x^2 - 12x = 0 \quad /:12$

$x(x-1) = 0 \rightarrow \underline{x_0 = 0 \wedge x_0 = 1}$

$y'' > 0$

$12x^2 - 12x > 0$

$x(x-1) > 0 \Leftrightarrow (x > 0 \wedge x > 1) \vee (x < 0 \wedge x < 1)$

$(1; \infty) \quad \cup \quad (-\infty; 0)$

$\Rightarrow$ $x_0 = 0$ a $x_0 = 1$ sú inflexné body funkcie

funkcia je na $(-\infty; 0)$ a na $(1; \infty)$ konvexná

funkcia je na $(0; 1)$ konkávna

c) $f: y = x^4 + 4x^3 - 18x^2 + 3x + 2$

$y' = 4x^3 + 12x^2 - 36x + 3$

$y'' = 12x^2 + 24x - 36 = 0 \quad /:12$

$x^2 + 2x - 3 = 0$

$D = 4 + 4 \cdot 3 = 16$

$x_{1,2} = \frac{-2 \pm 4}{2} = \begin{cases} -3 \\ 1 \end{cases} \Rightarrow \underline{x_0 = -3 \wedge x_0 = 1}$

$y'' > 0$

$x^2 + 2x - 3 > 0$

$(x+3)(x-1) > 0$

$(x > -3 \wedge x > 1) \vee (x < -3 \wedge x < 1)$

$(1; \infty) \quad \cup \quad (-\infty; -3)$

$\Rightarrow$ inflexné body: $x_0 = -3$ a $x_0 = 1$

konvexná na $(-\infty; -3)$ a na $(1; \infty)$

konkávna na $(-3; 1)$

6.1 

d) $f: y = x^4 - x^5$

$y' = 4x^3 - 5x^4$

$y'' = 12x^2 - 20x^3 = 0 \quad /:4$

$x^2 \cdot (3 - 5x) = 0 \Leftrightarrow x^2 = 0 \vee 3 - 5x = 0$

$\underline{x_0 = 0 \vee x_0 = \frac{3}{5}}$

$x^2 \cdot (3 - 5x) > 0 \Leftrightarrow 3 - 5x > 0 \Leftrightarrow x < \frac{3}{5} \quad [x^2 > 0 \text{ vždy}]$

$\Rightarrow$ inflexné body: $x_0 = 0 \wedge x_0 = \frac{3}{5}$

konvexná na $(-\infty; 0)$ a na $(0; \frac{3}{5})$

konkávna na $(\frac{3}{5}; \infty)$

e) $f: y = \frac{x^6}{6} - \frac{x^5}{4} + 3$

$y' = \frac{1}{6} \cdot 6x^5 - \frac{1}{4} \cdot 5x^4 = x^5 - \frac{5}{4}x^4$

$y'' = 5x^4 - \frac{5}{4} \cdot 4x^3 = 5x^4 - 5x^3 = 5x^3(x-1) = 0 \Leftrightarrow \underline{x_0 = 0 \vee x_0 = 1}$

$5x^3(x-1) > 0 \Leftrightarrow \underbrace{5x^2}_{>0} \cdot x \cdot (x-1) > 0 \Leftrightarrow x(x-1) > 0$

$(x > 0 \wedge x > 1) \vee (x < 0 \wedge x < 1)$

$(1; \infty) \quad \cup \quad (-\infty; 0)$

$\Rightarrow$ ib: $x_0 = 0 \wedge x_0 = 1$

konvexná na $(-\infty; 0)$ a na $(1; \infty)$

konkávna na $(0; 1)$

f) $f: y = 3x - (4-x)^5$

$y'' = (y')' = [3 - 5 \cdot (4-x)^4 \cdot (-1)]' = (3 + 5 \cdot (4-x)^4)' = 20 \cdot (4-x)^3 \cdot (-1) = -20 \cdot (4-x)^3 = 0 \Leftrightarrow \underline{x_0 = 4}$

$y'' > 0 \Leftrightarrow -20(4-x)^3 > 0 \Leftrightarrow \underbrace{-20 \cdot (4-x)^2}_{>0} \cdot (4-x) > 0 \Leftrightarrow -(4-x) > 0 \Leftrightarrow x - 4 > 0 \Leftrightarrow \underline{x > 4}$

$\Rightarrow$ ib: $x_0 = 4$

konvexná na $(4; \infty)$

konkávna na $(-\infty; 4)$

6.1 

g) $f: y = x^4 + 2x^3 + 6x^2$

$$y'' = (4x^3 + 6x^2 + 12x)' = 12x^2 + 12x + 12 = 0$$

$$x^2 + x + 1 = 0$$

$$D = 1 - 4 = -3 \Rightarrow \nexists \text{ inflexný bod}$$

postupujeme tak, že dosadíme do vzťahu ľubovoľné číslo a zistíme, aké znamienko má výsledok; platí: ak $D<0 \Rightarrow$ funkcia je vždy $>0$ alebo $<0$, teda aké znamienko bude mať výsledok, také znamienko má funkcia po dosadení všetkých $x \in D(f)$:

$$x^2 + x + 1 \to \text{dosadíme } \underline{0}: 0^2 + 0 + 1 = 1 > 0$$

$\Rightarrow y'' > 0 \ \forall x \in D(f) \Rightarrow$ ib $\nexists$

funkcia je konvexná na $(-\infty; \infty)$

h) $f: y = x + \frac{1}{x}$ ; $\boxed{D(f) = (-\infty; 0) \cup (0; \infty)}$

$$y'' = (1 - x^{-2})' = 2x^{-3} = \frac{2}{x^3} = 0 \to \text{nenastane nikdy}$$

$\Rightarrow \nexists$ ib

$$\frac{2}{x^3} > 0 \Leftrightarrow x^3 > 0 \Leftrightarrow \underline{x > 0}$$

$\Rightarrow$ ib $\nexists$

konvexná na $(0; \infty)$

konkávna na $(-\infty; 0)$

i) $f: y = 3x + \frac{1}{2x^2}$ ; $\boxed{D(f) = (-\infty; 0) \cup (0; \infty)}$

$$y'' = \left(3x + \frac{1}{2} \cdot x^{-2}\right)'' = \left(3 + \frac{1}{2} \cdot (-2)x^{-3}\right)' = (3 - x^{-3})' = 3x^{-4} = \frac{3}{x^4} \neq 0 \text{ nikdy} ; \ \frac{3}{x^4} > 0 \text{ vždy} \Rightarrow$$

ib $\nexists$

konvexná na celom $D(f)$

j) $f: y = \frac{3x^2}{1-x}$ ; $\boxed{D(f) = (-\infty; 1) \cup (1; \infty)}$

$$y' = \frac{6x \cdot (1-x) - 3x^2 \cdot (-1)}{(1-x)^2} = \frac{6x - 6x^2 + 3x^2}{(1-x)^2} = \frac{6x - 3x^2}{(1-x)^2}$$

$$y'' = \frac{(6-6x) \cdot (1-x)^2 - (6x - 3x^2) \cdot 2 \cdot (1-x) \cdot (-1)}{((1-x)^2)^2} = \frac{6 \cdot (1-x)^3 - 3x(2-x) \cdot (-2) \cdot (1-x)}{(1-x)^4} = \frac{(1-x) \cdot [6(1-x)^2 + 6(2-x)]}{(1-x)^4} = \circledast$$

$$(*) = \frac{6\cdot(1-2x+x^2+2-x)}{(1-x)^3} = \frac{6\cdot(x^2-3x+3)}{(1-x)^3} \neq 0 \text{ nikdy, lebo } x^2-3x+3\neq 0\text{, lebo } D=9-4\cdot 3<0$$

dosadíme $\underline{0}$: $0^2-3\cdot 0+3>0 \quad \forall x\in D(f)$

$$\frac{6(x^2-3x+3)}{(1-x)^3}>0 \Leftrightarrow \underbrace{\frac{\overset{>0}{6}(\overset{>0}{x^2-3x+3})}{(1-x)^2}}_{>0}\cdot\frac{1}{1-x}>0 \Leftrightarrow \frac{1}{1-x}>0$$

$$\Leftrightarrow 1-x>0 \Leftrightarrow \underline{x<1}$$

$\Rightarrow$ ib $\nexists$

konvexná na $(-\infty;1)$

konkávna na $(1;\infty)$

6.1 k) $f: y=\dfrac{x^2+x+21}{x+2}$ ; $\boxed{D(f)=(-\infty;-2)\cup(-2;\infty)}$

$$y'=\frac{(2x+1)(x+2)-(x^2+x+21)\cdot 1}{(x+2)^2}=\frac{2x^2+4x+x+2-x^2-x-21}{(x+2)^2}=\frac{x^2+4x-19}{(x+2)^2}$$

$$y''=\frac{(2x+4)\cdot(x+2)^2-(x^2+4x-19)\cdot 2(x+2)}{(x+2)^4}=\frac{2\cdot(x+2)^3-2(x^2+4x-19)(x+2)}{(x+2)^4}=\frac{2(x+2)\cdot[(x+2)^2-(x^2+4x-19)]}{(x+2)^4}=$$

$$=\frac{2\cdot(x^2+4x+4-x^2-4x+19)}{(x+2)^3}=\frac{46}{(x+2)^3}\neq 0 \text{ nikdy; } \frac{46}{(x+2)^3}>0 \Leftrightarrow \underbrace{\frac{46}{(x+2)^2}}_{>0}\cdot\frac{1}{x+2}>0 \Leftrightarrow x+2>0$$

$$\underline{x>-2}$$

$\Rightarrow$ ib $\nexists$

konvexná na $(-2;\infty)$

konkávna na $(-\infty;-2)$

6.1

l) $f: y=\frac{2x}{1+x^2}$ ; $\boxed{D(f)=\mathbb{R}}$

$$y'=\frac{2\cdot(1+x^2)-2x\cdot 2x}{(1+x^2)^2}=\frac{2+2x^2-4x^2}{(1+x^2)^2}=\frac{2-2x^2}{(1+x^2)^2}$$

$$y''=\frac{(-4x)(1+x^2)^2-(2-2x^2)\cdot 2\cdot(1+x^2)\cdot 2x}{(1+x^2)^4}=\frac{(1+x^2)\cdot[-4x(1+x^2)-4x(2-2x^2)]}{(1+x^2)^4}=\frac{-4x-4x^3-8x+8x^3}{(1+x^2)^3}=\frac{4x^3-12x}{(1+x^2)^3}=$$

$$=\frac{4x(x^2-3)}{(1+x^2)^3}=0 \Leftrightarrow x_0=0 \wedge x^2-3=0 \quad x_0=\pm\sqrt{3}$$

$$\frac{4x(x^2-3)}{(1+x^2)^3}>0 \Leftrightarrow \underbrace{\frac{4}{(1+x^2)^3}}_{>0}\cdot x(x^2-3)>0 \Leftrightarrow$$

$$\Leftrightarrow x(x-\sqrt{3})(x+\sqrt{3})>0$$

| | $(-\infty;-\sqrt{3})$ | $(-\sqrt{3};0)$ | $(0;\sqrt{3})$ | $(\sqrt{3};\infty)$ |
|---|---|---|---|---|
| $x$ | $-$ | $-$ | $+$ | $+$ |
| $x-\sqrt{3}$ | $-$ | $-$ | $-$ | $+$ |
| $x+\sqrt{3}$ | $-$ | $+$ | $+$ | $+$ |
| | $-$ | $+$ | $-$ | $+$ |

$\Rightarrow$ ib sú $x_0=0 \wedge x_0=-\sqrt{3} \wedge x_0=\sqrt{3}$

konvexná na $(-\sqrt{3};0)$ a na $(\sqrt{3};\infty)$

konkávna na $(-\infty;-\sqrt{3})$ a na $(0;\sqrt{3})$

6.1

m) $f: y = \dfrac{x}{1-x^2}$ ; $\boxed{D(f) = (-\infty; -1) \cup (-1; 1) \cup (1; \infty)} = \mathbb{R} \setminus \{-1; 1\}$

$$y' = \frac{1\cdot(1-x^2) - x\cdot(-2x)}{(1-x^2)^2} = \frac{1-x^2+2x^2}{(1-x^2)^2} = \frac{x^2+1}{(1-x^2)^2}$$

$$y'' = \frac{2x\cdot(1-x^2)^2 - (x^2+1)\cdot 2\cdot(1-x^2)\cdot(-2x)}{(1-x^2)^4} = \frac{(1-x^2)\cdot[2x(1-x^2)+4x(x^2+1)]}{(1-x^2)^4} = \frac{2x-2x^3+4x^3+4x}{(1-x^2)^3} =$$

$$= \frac{2x^3+6x}{(1-x^2)^3} = \frac{2x(x^2+3)}{(1-x^2)^3} = \underbrace{\frac{2(x^2+3)}{(1-x^2)^2}}_{>0}\cdot\frac{x}{1-x^2} = 0 \Leftrightarrow \underline{x_0 = 0}$$

$$\frac{2(x^2+3)}{(1-x^2)^2}\cdot\frac{x}{(1-x^2)} > 0 \Leftrightarrow \frac{x}{(1-x^2)} > 0 \Leftrightarrow \frac{x}{(1-x)(1+x)} > 0$$

| | $(-\infty; -1)$ | $(-1; 0)$ | $(0; 1)$ | $(1; \infty)$ |
|---|---|---|---|---|
| $x$ | $-$ | $-$ | $+$ | $+$ |
| $1-x$ | $+$ | $+$ | $+$ | $-$ |
| $1+x$ | $-$ | $+$ | $+$ | $+$ |
| | $+$ | $-$ | $+$ | $-$ |

$\Rightarrow$ ib je $x_0 = 0$

konvexná na $(-\infty; -1)$ a na $(0; 1)$

konkávna na $(-1; 0)$ a na $(1; \infty)$

6.1 

m) $f: y = \frac{x^2}{16-x^2}$ ; $\boxed{D(f) = \mathbb{R} \setminus \{-4; 4\}}$

$$y' = \frac{2x(16-x^2) - x^2(-2x)}{(16-x^2)^2} = \frac{32x - 2x^3 + 2x^3}{(16-x^2)^2} = \frac{32x}{(16-x^2)^2}$$

$$y'' = \frac{32(16-x^2)^2 - 32x(16-x^2)\cdot 2\cdot(-2x)}{(16-x^2)^4} = \frac{(16-x^2)\cdot[32\cdot(16-x^2) + 32\cdot 4x^2]}{(16-x^2)^4} = \frac{32\cdot(16-x^2+4x^2)}{(16-x^2)^3} = \frac{32\cdot(3x^2+16)}{(16-x^2)^3} =$$

$$= \frac{32\cdot(3x^2+16)}{(16-x^2)^3} = 0 \Leftrightarrow 3x^2+16=0 \text{, ale } 3x^2+16>0 \text{ vždy}$$

$$\frac{32(3x^2+16)}{(16-x^2)^3} = \underbrace{\frac{32(3x^2+16)}{(16-x^2)^2}}_{>0} \cdot \frac{1}{16-x^2} > 0 \Leftrightarrow \frac{1}{16-x^2} > 0 \Leftrightarrow (4-x)(4+x) > 0$$

$$(4-x>0 \wedge 4+x>0) \vee (4-x<0 \wedge 4+x<0)$$

$$(x<4 \wedge x>-4) \vee (x>4 \wedge x<-4)$$

$$(-4; 4) \qquad \emptyset$$

$\Rightarrow$ ib $\nexists$

konvexná na $(-4; 4)$

konkávna na $(-\infty; -4)$ a na $(4; \infty)$

6.2 a) $f: y = \frac{x^2+1}{x^2-1}$ ; $\boxed{D(f) = \mathbb{R} \setminus \{-1; 1\}}$

$$y' = \frac{2x(x^2-1) - (x^2+1)\cdot 2x}{(x^2-1)^2} = \frac{2x\cdot[x^2-1-x^2-1]}{(x^2-1)^2} = \frac{-4x}{(x^2-1)^2}$$

$$y'' = \frac{-4\cdot(x^2-1)^2 - (-4x)\cdot 2(x^2-1)\cdot 2x}{(x^2-1)^4} = \frac{(x^2-1)\cdot[-4(x^2-1)+16x^2]}{(x^2-1)^4} = \frac{-4x^2+4+16x^2}{(x^2-1)^3} = \frac{12x^2+4}{(x^2-1)^3} = 0 \Leftrightarrow 12x^2+4=0,\ \text{ale } 12x^2+4>0 \text{ vždy}$$

$$\frac{12x^2+4}{(x^2-1)^3} > 0 \Leftrightarrow \underbrace{\frac{12x^2+4}{(x^2-1)^2}}_{>0} \cdot \frac{1}{x^2-1} > 0 \Leftrightarrow \frac{1^{>0}}{x^2-1} > 0 \Leftrightarrow x^2-1>0 \Leftrightarrow (x-1)(x+1)>0$$

$$(x>1 \wedge x>-1) \vee (x<1 \wedge x<-1)$$

$$(1;\infty) \quad \cup \quad (-\infty;-1)$$

$\Rightarrow$ ib $\nexists$

konvexná na $(-\infty;-1)$ a na $(1;\infty)$

konkávna na $(-1;1)$

b) $f: y = \frac{1}{x^3} + \frac{1}{x^2}$ ; $\boxed{D(f) = \mathbb{R} \setminus \{0\}}$

$$y'' = (x^{-3} + x^{-2})'' = (-3x^{-4} - 2x^{-3})' = 12x^{-5} + 6x^{-4} = \frac{12}{x^5} + \frac{6}{x^4} = \frac{12+6x}{x^5} = 0 \Leftrightarrow 12+6x=0 \Leftrightarrow x_0 = -2$$

$$\frac{12+6x}{x^5} > 0 \Leftrightarrow \underbrace{\frac{1}{x^4}}_{>0} \cdot \frac{12+6x}{x} > 0 \Leftrightarrow \frac{12+6x}{x} > 0 \Leftrightarrow (12+6x>0 \wedge x>0) \vee (12+6x<0 \wedge x<0)$$

$$(x>-2 \wedge x>0) \vee (x<-2 \wedge x<0)$$

$$(0;\infty) \quad \cup \quad (-\infty;-2)$$

$\Rightarrow$ ib je $x_0 = -2$

konvexná na $(-\infty;-2)$ a na $(0;\infty)$ ; konkávna na $(-2;0)$

c) $f: y = \frac{1}{x^3} - \frac{6}{x}$ ; $\boxed{D(f) = \mathbb{R} \setminus \{0\}}$

$$y'' = (x^{-3} - 6x^{-1})'' = (-3x^{-4} + 6x^{-2})' = 12x^{-5} - 12x^{-3} = \frac{12}{x^5} - \frac{12}{x^3} = \frac{12-12x^2}{x^5} = 0 \Leftrightarrow 12 - 12x^2 = 0 \Leftrightarrow x^2 = 1$$

$x_0 = -1 \wedge x_0 = 1$

$$\frac{12-12x^2}{x^5} > 0 \Leftrightarrow \underbrace{\frac{12}{x^4}}_{>0} \cdot \frac{1-x^2}{x} > 0 \Leftrightarrow \frac{(1-x)(1+x)}{x} > 0$$

| | $(-\infty; -1)$ | $(-1; 0)$ | $(0; 1)$ | $(1; \infty)$ |
|---|---|---|---|---|
| $x$ | − | − | + | + |
| $1-x$ | + | + | + | − |
| $1+x$ | − | + | + | + |
| | + | − | + | − |

$\Rightarrow$ ib je $x_0 = 1 \wedge x_0 = -1$

konvexná na $(-\infty; -1)$ a na $(0; 1)$

konkávna na $(-1; 0)$ a na $(1; \infty)$

d) $f: y = \left(\frac{1}{2} + \frac{1}{x}\right)^2$ ; $\boxed{D(f) = \mathbb{R} \setminus \{0\}}$

$$y' = \left[\left(\frac{x+2}{2x}\right)^2\right]' = 2 \cdot \frac{x+2}{2x} \cdot \frac{1 \cdot 2x - (x+2) \cdot 2}{(2x)^2} = \frac{2(x+2)(2x-2x-4)}{(2x)^3} = \frac{-8(x+2)}{(2x)^3} = \frac{-8x-16}{(2x)^3}$$

$$y'' = \frac{-8(2x)^3 - (-8x-16) \cdot 3 \cdot (2x)^2 \cdot 2}{[(2x)^3]^2} = \frac{(2x)^2 \cdot [-8 \cdot 2x + 6 \cdot (8x+16)]}{(2x)^6} = \frac{-16x + 48x + 96}{(2x)^4} = \frac{32x + 96}{(2x)^4} = 0 \Leftrightarrow 32x = -96$$

$x_0 = -3$

$$\frac{32x+96}{(2x)^4} = \underbrace{\frac{32}{(2x)^4}}_{>0} \cdot \frac{x+3}{1} > 0 \Leftrightarrow x + 3 > 0 \Leftrightarrow x > -3 \Rightarrow$$

ib je $x_0 = -3$

konvexná na $(-3; 0)$ a na $(0; \infty)$

konkávna na $(-\infty; -3)$

e) $f: y = 3x - \sqrt{x-3}$ ; $D(f)$: $x-3 \geq 0$

$x \geq 3 \Rightarrow \boxed{D(f) = \langle 3; \infty)}$

$y'' = \left(3 - \frac{1}{2}(x-3)^{-\frac{1}{2}}\right)' = \frac{1}{4}(x-3)^{-\frac{3}{2}} = \frac{1}{4\cdot\sqrt{(x-3)^3}} > 0$ vždy $\Rightarrow$ $\nexists$ ib, konvexná na $\langle 3; \infty)$

$$a^x \cdot b^x = (a\cdot b)^x$$
$$\frac{a^x}{b^x} = \left(\frac{a}{b}\right)^x$$
$$(a^x)^y = a^{x\cdot y}$$
$$a^x \cdot a^y = a^{x+y}$$
$$\frac{a^x}{a^y} = a^{x-y}$$

f) $f: y = 4 + \sqrt[3]{x^2}$ ; $\boxed{D(f) = \mathbb{R}}$

$y'' = \left(\frac{2}{3}x^{-\frac{1}{3}}\right)' = -\frac{2}{9}x^{-\frac{4}{3}} = \frac{-2}{9\cdot\sqrt[3]{x^4}} \rightsquigarrow$ $\left.\begin{array}{l}\text{čitateľ } < 0 \text{ vždy}\\ \text{menovateľ } > 0 \text{ vždy}\end{array}\right\} \Rightarrow \frac{-2}{9\cdot\sqrt[3]{x^4}} < 0$ vždy $\Rightarrow$ $\nexists$ ib, konkávna na $(-\infty; \infty)$

g) $f: y = \frac{2x}{\sqrt{x^2+1}}$ ; $\boxed{D(f) = \mathbb{R}}$

$$y' = \frac{2\cdot(x^2+1)^{\frac{1}{2}} - 2x\cdot\frac{1}{2}(x^2+1)^{-\frac{1}{2}}\cdot 2x}{\left(\sqrt{x^2+1}\right)^2} = \frac{2\cdot\sqrt{x^2+1} - 2x^2\cdot\frac{1}{\sqrt{x^2+1}}}{x^2+1} = \frac{\frac{2\cdot(x^2+1) - 2x^2}{\sqrt{x^2+1}}}{x^2+1} = \frac{2x^2+2-2x^2}{(x^2+1)\cdot\sqrt{x^2+1}} = \frac{2}{(x^2+1)\cdot(x^2+1)^{\frac{1}{2}}} =$$

$$= \frac{2}{(x^2+1)^{\frac{3}{2}}}$$

$$y'' = \frac{-2\cdot\frac{3}{2}\cdot(x^2+1)^{\frac{1}{2}}\cdot 2x}{\left[(x^2+1)^{\frac{3}{2}}\right]^2} = \frac{-6x\cdot(x^2+1)^{\frac{1}{2}}}{(x^2+1)^3} = \frac{-6x}{(x^2+1)^{\frac{5}{2}}} = \frac{-6}{(x^2+1)^2}\cdot\frac{x}{(x^2+1)^{\frac{1}{2}}} = \frac{-6}{(x^2+1)^2\cdot\sqrt{x^2+1}}\cdot x = 0 \Leftrightarrow \underline{x_0 = 0}$$

$\frac{\overset{<0}{-6}}{\underset{>0}{(x^2+1)^2}\cdot\underset{>0}{\sqrt{x^2+1}}}\cdot x > 0 \Leftrightarrow x < 0$, lebo [záporné č.]$\cdot x > 0 \Leftrightarrow x < 0 \Rightarrow$ ib je $x_0 = 0$

na $(-\infty; 0)$ je konvexná

na $(0; \infty)$ je konkávna

(6.2) 

h) $f: y = \frac{x}{\sqrt{x^3+1}}$ ; $D(f): \underbrace{x^3+1 \geq 0 \wedge \sqrt{x^3+1} \neq 0}_{x^3+1>0}$

$x^3 > -1 \Leftrightarrow x > -1 \Rightarrow \boxed{D(f) = (-1;\infty)}$

$$y' = \frac{1 \cdot \frac{1}{2}(x^3+1)^{\frac{1}{2}} - x \cdot \frac{1}{2}(x^3+1)^{-\frac{1}{2}} \cdot 3x^2}{x^3+1} = \frac{\frac{1}{2}(x^3+1)^{\frac{1}{2}} - \frac{3}{2}x^3(x^3+1)^{-\frac{1}{2}}}{x^3+1} = \frac{\frac{\frac{1}{2}\cdot(x^3+1) - \frac{3}{2}x^3}{\sqrt{x^3+1}}}{x^3+1} = \frac{-x^3+\frac{1}{2}}{(x^3+1)^{\frac{3}{2}}}$$

$$y'' = \frac{(-3x^2)(x^3+1)^{\frac{3}{2}} - (-x^3+\frac{1}{2})\cdot\frac{3}{2}\cdot(x^3+1)^{\frac{1}{2}}\cdot 3x^2}{\left[(x^3+1)^{\frac{3}{2}}\right]^2} = \frac{(x^3+1)^{\frac{1}{2}}\cdot\left[-3x^2\cdot(x^3+1) - \frac{9}{2}x^2\cdot(-x^3+\frac{1}{2})\right]}{(x^3+1)^3} = \frac{-3x^5-3x^2+\frac{9}{2}x^5-\frac{9}{4}x^2}{(x^3+1)^{\frac{5}{2}}} =$$

$$= \frac{x^5\cdot(-\frac{6}{2}+\frac{9}{2}) + x^2\cdot(-\frac{12}{4}-\frac{9}{4})}{(x^3+1)^{\frac{5}{2}}} = \frac{\frac{3}{2}x^5 - \frac{21}{4}x^2}{(x^3+1)^{\frac{5}{2}}} = \frac{\frac{3}{4}x^2\cdot(2x^3-7)}{(x^3+1)^2\cdot\sqrt{x^3+1}} = \frac{\overset{\geq 0}{\frac{3}{4}x^2}}{\underset{>0}{(x^3+1)^2}\cdot\underset{>0}{\sqrt{x^3+1}}}\cdot(2x^3-7) = 0 \Leftrightarrow$$

$\Leftrightarrow x^2 = 0 \vee 2x^3-7=0$

$\boxed{x_0 = 0} \vee x^3 = \frac{7}{2}$

$\boxed{x_0 = \sqrt[3]{\frac{7}{2}}}$

$y'' > 0 \Leftrightarrow 2x^3 - 7 > 0$

$x^3 > \frac{7}{2}$

$\boxed{x > \sqrt[3]{\frac{7}{2}}}$

$\Rightarrow$ ib sú $x_0 = 0 \wedge x_0 = \sqrt[3]{\frac{7}{2}}$

konvexná na $(\sqrt[3]{\frac{7}{2}};\infty)$

konkávna na $(-1;0) \cup (0;\sqrt[3]{\frac{7}{2}})$

6.3 

a) $f: y = x\cdot e^{-x}$

$$y'' = (x\cdot e^{-x})'' = \left[1\cdot e^{-x} + x\cdot e^{-x}\cdot(-1)\right]' = \left[e^{-x}\cdot(1-x)\right]' = e^{-x}\cdot(-1)\cdot(1-x) + e^{-x}\cdot(-1) = e^{-x}\cdot\left(-1\cdot(1-x)-1\right) = e^{-x}(-1+x-1) = e^{-x}\cdot(x-2) = 0 \Leftrightarrow$$

$$\Leftrightarrow \underline{x_0 = 2}$$

$$y'' > 0 \Leftrightarrow \frac{x-2}{\underbrace{e^{x}}_{>0}} > 0 \Leftrightarrow x-2>0 \Leftrightarrow x>2 \Rightarrow$$

ib je $x_0 = 2$

konvexná na $(2;\infty)$

konkávna na $(-\infty;2)$

b) $f: y = e^{-x^2}$

$$y'' = \left[e^{-x^2}\cdot(-2x)\right]' = e^{-x^2}\cdot(-2x)\cdot(-2x) + e^{-x^2}\cdot(-2) = e^{-x^2}\cdot\left((-2x)^2-2\right) = \frac{4x^2-2}{e^{x^2}} = 0 \Leftrightarrow 4x^2-2=0 \Leftrightarrow x^2 = \frac{1}{2}$$

$$\underline{x_0 = \pm\sqrt{\tfrac{1}{2}}}$$

$$y'' > 0 \Leftrightarrow \frac{4x^2-2}{\underbrace{e^{x^2}}_{>0}} > 0 \Leftrightarrow 4x^2-2>0$$

$$4x^2 > 2$$

$$x^2 > \frac{1}{2} \Rightarrow (-\infty;-\sqrt{\tfrac{1}{2}}) \cup (\sqrt{\tfrac{1}{2}};\infty)$$

$$(x-\sqrt{\tfrac{1}{2}})(x+\sqrt{\tfrac{1}{2}})>0$$

...

$\Rightarrow$ ib je $x_0 = -\sqrt{\tfrac{1}{2}} \wedge x_0 = \sqrt{\tfrac{1}{2}}$

konvexná na $(-\infty;-\sqrt{\tfrac{1}{2}})$ a na $(\sqrt{\tfrac{1}{2}};\infty)$

konkávna na $(-\sqrt{\tfrac{1}{2}};\sqrt{\tfrac{1}{2}})$

c) $f: y = x\cdot e^{-x^2}$

$$y' = 1\cdot e^{-x^2} + x\cdot e^{-x^2}\cdot(-2x) = e^{-x^2}\cdot(1-2x^2)$$

$$y'' = e^{-x^2}\cdot(-2x)\cdot(1-2x^2) + e^{-x^2}\cdot(-4x) = e^{-x^2}\cdot\left(-2x\cdot(1-2x^2)-4x\right) = e^{-x^2}\cdot(-2x+4x^3-4x) = e^{-x^2}\cdot(4x^3-6x) = \frac{4x^3-6x}{e^{x^2}} = 0 \Leftrightarrow$$

$$\Leftrightarrow 4x^3-6x=0 \Leftrightarrow 2x(2x^2-3)=0 \Leftrightarrow \underline{x_0=0} \vee$$

$$\vee\ 2x^2=3 \Leftrightarrow \underline{x_0 = \pm\sqrt{\tfrac{3}{2}}}$$

c) $y'' > 0 \Leftrightarrow \dfrac{4x^3-6x}{\underbrace{e^{x^2}}_{>0}} > 0 \Leftrightarrow 4x^3-6x>0 \Leftrightarrow 2x(2x^2-3)>0 \Leftrightarrow 4x\left(x^2-\frac{3}{2}\right)>0 \Leftrightarrow x\cdot\left(x-\sqrt{\frac{3}{2}}\right)\cdot\left(x+\sqrt{\frac{3}{2}}\right)>0$

| | $\left(-\infty;-\sqrt{\frac{3}{2}}\right)$ | $\left(-\sqrt{\frac{3}{2}};0\right)$ | $\left(0;\sqrt{\frac{3}{2}}\right)$ | $\left(\sqrt{\frac{3}{2}};\infty\right)$ |
|---|---|---|---|---|
| $x$ | $-$ | $-$ | $+$ | $+$ |
| $x-\sqrt{\frac{3}{2}}$ | $-$ | $-$ | $-$ | $+$ |
| $x+\sqrt{\frac{3}{2}}$ | $-$ | $+$ | $+$ | $+$ |
| | $-$ | $+$ | $-$ | $+$ |

$\Leftarrow$

ib sú $x_0=0$

$x_0=-\sqrt{\frac{3}{2}}$

$x_0=\sqrt{\frac{3}{2}}$

konvexná na $\left(-\sqrt{\frac{3}{2}};0\right)$

a na $\left(\sqrt{\frac{3}{2}};\infty\right)$

konkávna na $\left(-\infty;-\sqrt{\frac{3}{2}}\right)$

a na $\left(0;\sqrt{\frac{3}{2}}\right)$

d) $f: y=x^2\cdot e^{-x}$

$y' = 2x\cdot e^{-x} + x^2\cdot e^{-x}\cdot(-1) = e^{-x}\cdot(2x-x^2)$

$y'' = e^{-x}\cdot(-1)\cdot(2x-x^2) + e^{-x}\cdot(2-2x) = e^{-x}\cdot(x^2-2x+2-2x) = e^{-x}\cdot(x^2-4x+2) = \dfrac{x^2-4x+2}{e^x} = 0 \Leftrightarrow x^2-4x+2=0$

$D = 16-4\cdot 2 = 8$

$x_{1,2} = \dfrac{4\pm\sqrt{8}}{2} = \dfrac{4\pm\sqrt{4\cdot 2}}{2} = \dfrac{4\pm 2\cdot\sqrt{2}}{2} = \dfrac{\cancel{2}\cdot(2\pm\sqrt{2})}{\cancel{2}} = 2\pm\sqrt{2}$

$\Rightarrow x_0 = 2-\sqrt{2} \wedge x_0 = 2+\sqrt{2}$

$y''>0 \Leftrightarrow \dfrac{x^2-4x+2}{\underbrace{e^x}_{>0}} > 0 \Leftrightarrow (x-2+\sqrt{2})(x-2-\sqrt{2})>0$

$(x>2-\sqrt{2} \wedge x>2+\sqrt{2}) \vee (x<2-\sqrt{2} \wedge x<2+\sqrt{2})$

$(2+\sqrt{2};\infty) \quad \cup \quad (-\infty;2-\sqrt{2})$

$\Rightarrow$ ib sú $x_0=2-\sqrt{2}$ a $x_0=2+\sqrt{2}$

konvexná na $(-\infty;2-\sqrt{2})$ a na $(2+\sqrt{2};\infty)$

konkávna na $(2-\sqrt{2};2+\sqrt{2})$

6.3

e) $f: y = x\cdot e^{\frac{1}{x}}$; $\boxed{D(f)=\mathbb{R}\setminus\{0\}}$

$$y'' = \left[1\cdot e^{\frac{1}{x}} + x\cdot e^{\frac{1}{x}}\cdot(-x^{-2})\right]' = \left[e^{\frac{1}{x}}\cdot(1-x^{-1})\right]' = e^{\frac{1}{x}}\cdot(-x^{-2})(1-x^{-1}) + e^{\frac{1}{x}}\cdot x^{-2} = e^{\frac{1}{x}}\cdot\left(\frac{1-\frac{1}{x}}{-x^2}+\frac{1}{x^2}\right) = e^{\frac{1}{x}}\cdot\frac{\frac{1}{x}-1+1}{x^2} =$$

$$= e^{\frac{1}{x}}\cdot\frac{1}{x^3} = \frac{\overbrace{e^{\frac{1}{x}}}^{>0}}{x^3} \neq 0 \text{ nikdy}$$

ib ∄

$$y''>0 \Leftrightarrow \frac{e^{\frac{1}{x}}}{x^3}>0 \Leftrightarrow x^3>0 \Leftrightarrow x>0 \Rightarrow$$

konvexná na $(0;\infty)$
konkávna na $(-\infty;0)$

f) $f: y = e^{4-\frac{x^2}{2}}$

$$y'' = \left[e^{4-\frac{x^2}{2}}\cdot\left(-\tfrac{1}{2}\cdot 2x\right)\right]' = \left(-x\cdot e^{4-\frac{x^2}{2}}\right)' = -1\cdot e^{4-\frac{x^2}{2}} + (-x)\cdot e^{4-\frac{x^2}{2}}\cdot(-x) = e^{4-\frac{x^2}{2}}\cdot(-1+x^2) = (x^2-1)\cdot e^{4-\frac{x^2}{2}} = 0 \Leftrightarrow$$

$$\Leftrightarrow x^2=1 \Rightarrow \underline{x_0=-1 \wedge x_0=1}$$

$$y''>0 \Leftrightarrow (x^2-1)\cdot \underbrace{e^{4-\frac{x^2}{2}}}_{>0}>0 \Leftrightarrow x^2-1>0 \Leftrightarrow x^2>1 \Rightarrow (-\infty;-1)\cup(1;\infty)$$

$(x+1)(x-1)>0$

$\Rightarrow$ ib sú $x_0=-1$ a $x_0=1$
konvexná na $(-\infty;-1)$ a na $(1;\infty)$
konkávna na $(-1;1)$

g) $f: y = e^{1-\frac{x^3}{3}}$

$$y'' = \left[e^{1-\frac{x^3}{3}}\cdot\left(-\tfrac{1}{3}\cdot 3x^2\right)\right]' = \left(e^{1-\frac{x^3}{3}}\cdot(-x^2)\right)' = e^{1-\frac{x^3}{3}}\cdot(-x^2)\cdot(-x^2) + e^{1-\frac{x^3}{3}}\cdot(-2x) = e^{1-\frac{x^3}{3}}\cdot(x^4-2x) = x\cdot \underbrace{e^{1-\frac{x^3}{3}}}_{>0}\cdot(x^3-2) = 0 \Leftrightarrow$$

$$\Leftrightarrow x(x^3-2)=0 \Leftrightarrow \underline{x_0=0} \vee x^3=2$$

$\underline{x_0=\sqrt[3]{2}}$

$$x e^{1-\frac{x^3}{3}}\cdot(x^3-2)>0 \Leftrightarrow x(x^3-2)>0 \Leftrightarrow (x>0 \wedge x>\sqrt[3]{2}) \vee (x<0 \wedge x<\sqrt[3]{2})$$

$(\sqrt[3]{2};\infty) \cup (-\infty;0)$

g) ib sú $x_0 = 0 \wedge x_0 = \sqrt[3]{2}$

konvexná na $(-\infty; 0)$ a na $(\sqrt[3]{2}; \infty)$

konkávna na $(0; \sqrt[3]{2})$

h) $f: y = e^{2x} - 8e^x + 5x$

$$y'' = [e^{2x} \cdot 2 - 8e^x + 5]' = 4e^{2x} - 8e^x = \underbrace{4e^x}_{>0} \cdot (e^x - 2) = 0 \Leftrightarrow e^x = 2$$

$\boxed{x_0 = \ln 2}$

$4e^x(e^x - 2) > 0 \Leftrightarrow e^x > 2 \Leftrightarrow x > \ln 2 \Rightarrow$

ib je $x_0 = \ln 2$

konvexná na $(\ln 2; \infty)$

konkávna na $(-\infty; \ln 2)$

i) $f: y = (2 - x^2) \cdot e^{-x}$

$$y'' = [-2x \cdot e^{-x} + (2 - x^2) \cdot e^{-x} \cdot (-1)]' = [e^{-x} \cdot (-2x - 2 + x^2)]' = [e^{-x}(x^2 - 2x - 2)]' = e^{-x} \cdot (-1) \cdot (x^2 - 2x - 2) + e^{-x} \cdot (2x - 2) =$$

$$= e^{-x} \cdot (-x^2 + 2x + 2 + 2x - 2) = e^{-x} \cdot (-x^2 + 4x) = x e^{-x}(4 - x) = 0 \Leftrightarrow \boxed{x_0 = 0} \wedge \boxed{x_0 = 4}$$

$$y'' > 0 \Leftrightarrow x(4 - x)\underset{>0}{e^{-x}} > 0 \Leftrightarrow (x > 0 \wedge x < 4) \vee (x < 0 \wedge x > 4) \Rightarrow$$

$(0; 4)$

ib sú $x_0 = 0$ a $x_0 = 4$

konvexná na $(0; 4)$

konkávna na $(-\infty; 0)$ a na $(4; \infty)$

6.3 j) $f: y=\frac{e^x}{x}$ ; $\boxed{D(f)=\mathbb{R}\setminus\{0\}}$ 

$$y''=\left(\frac{e^x\cdot x-e^x\cdot 1}{x^2}\right)'=\left(\frac{e^x\cdot(x-1)}{x^2}\right)'=\frac{[e^x\cdot(x-1)+e^x\cdot 1]\cdot x^2-e^x(x-1)\cdot 2x}{x^4}=\frac{x^2\cdot e^x\cdot(x-1+1)-2x\cdot e^x\cdot(x-1)}{x^4}=\frac{xe^x\cdot(x^2-2(x-1))}{x^4}=$$

$$=\frac{e^x\cdot(x^2-2x+2)}{x^3}=0 \Leftrightarrow x^2-2x+2=0$$

$D=4-4\cdot 2<0 \rightarrow$ dosadíme 0: $0^2-2\cdot 0+2=2>0 \Rightarrow x^2-2x+2>0$ vždy $(\forall x\in D(f))$

ib $\nexists$

$$y''>0 \Leftrightarrow \frac{\overset{>0}{e^x}\overbrace{(x^2-2x+2)}^{>0}}{x^3}>0 \Leftrightarrow x^3>0 \Leftrightarrow x>0 \Rightarrow$$

konvexná na $(0;\infty)$
konkávna na $(-\infty;0)$

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L'Hospitalovo pravidlo:

Ak $\lim\limits_{x\to a}\frac{f(x)}{g(x)}=\frac{\lim\limits_{x\to a}f(x)}{\lim\limits_{x\to a}g(x)}=\frac{0}{0}$ alebo $\frac{\pm\infty}{\pm\infty}$, potom $\boxed{\lim\limits_{x\to a}\frac{f(x)}{g(x)}=\lim\limits_{x\to a}\frac{f'(x)}{g'(x)}}$